

Study of continuous-time quantum walks on quotient graphs via quantum probability theory

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Abstract

In the present paper, we study the continuous-time quantum walk on quotient graphs. On such graphs, there is a straightforward reduction of problem to a subspace that can be considerably smaller than the original one. Along the lines of reductions, by using the idea of calculation of the probability amplitudes for continuous-time quantum walk in terms of the spectral distribution associated with the adjacency matrix of graphs [Jafarizadeh and Salimi (Ann. Phys 322(2007))], we show the continuous-time quantum walk on original graph Γ induces a continuous-time quantum walk on quotient graph Γ_H . Finally, for example we investigate continuous-time quantum walk on some quotient Cayley graphs.

Keywords: Continuous-time quantum walk, Spectral distribution, Cayley graph, Quotient graph.

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1 Introduction

Two types of quantum walks, discrete and continuous time, were introduced as the quantum mechanical extension of the corresponding random walks and have been extensively studied over the last few years [1, 2]. In recent years, quantum walk has gained considerable interest in the quantum information and computation research areas due to its potential applications. In particular, the study of continuous-time quantum walks (CTQW) on graphs has shown promising applications in the algorithmic and implementation aspects. As an alternate algorithmic technique to the Quantum Fourier Transform and the Amplitude Amplification techniques, Childs et al.[3] demonstrated the power of CTQW algorithm for solving a specific blackbox graph search problem. A study of quantum walks on simple graph is well known in physics(see [4]). Recent studies of quantum walks on more general graphs were described in [5, 6, 7, 8, 9, 10, 11, 12]. Some of these works studies the problem in the important context of algorithmic problems on graphs and suggests that quantum walks is a promising algorithmic technique for designing future quantum algorithms.

One approach for investigation of CTQW on graphs is using the spectral distribution associated with the adjacency matrix of graphs [13, 14, 15, 16, 17]. Authors in Ref.[13] have introduced a new method for calculating the probability amplitudes of quantum walk based on spectral distribution. The method of spectral distribution only requires simple structural data of graph and allows us to avoid a heavy combinational argument often necessary to obtain full description of spectrum of the adjacency matrix. In view of the fact that the quotient graphs are important to generate the Crystallographic nets, it is tempting to try to investigate the CTQW on quotient graphs. On such graphs, there is a straightforward reduction of problem to a subspace that can be considerably smaller than the original one. In this paper, along the lines of similar reductions and the idea calculation of the probability amplitudes for CTQW in terms of the spectral distribution associated with the adjacency matrix of graphs [13], we show

the CTQW on graph Γ reductions to CTQW on quotient graph Γ_H such that its adjacency matrix is contained two Szegő- Jacobi sequences $\sqrt{\omega_i}$ and α_i which denote the jumping rate from a vertex to its neighbor and self loops, respectively. Finally, we calculate the probability amplitude for CTQW on some quotient Cayley graphs.

The organization of the paper is as follows. In Section 2, we give a brief outline of Cayley graphs and automorphism groups. In Section 3, we investigate the action of groups and stratification, quantum decomposition for adjacency matrix of graphs, quotient graph, the method for obtaining spectral distribution μ and devoted to the method of computing the amplitude for CTQW, through spectral distribution μ of the adjacency matrix A . In the Section 4 we calculate the probability amplitude for CTQW on some quotient Cayley graphs. The paper is ended with a brief conclusion.

2 Cayley graphs and automorphism groups

A graph $\Gamma(V, E)$ consists of a non-empty vertex set V together with an edge set E that is a subset of $\{\{\alpha, \beta\} | \alpha, \beta \in V, \alpha \neq \beta\}$. Elements of V and of E are called *vertices* and *edges*, respectively. Two vertices $\alpha, \beta \in V$ are called adjacent if $\{\alpha, \beta\} \in E$, and in this case we write $\alpha \sim \beta$. Let $l^2(V)$ denote the Hilbert space of C-valued square-summable functions on V , and $\{|\alpha\rangle | \alpha \in V\}$ becomes a complete orthonormal basis of $l^2(V)$. The adjacency matrix $A = (A_{\alpha\beta})_{\alpha, \beta \in V}$ is defined by

$$A_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha \sim \beta \\ 0 & \text{otherwise.} \end{cases}$$

which is considered as an operator acting in $l^2(V)$ in such a way that

$$A|\alpha\rangle = \sum_{\alpha \sim \beta} |\beta\rangle, \quad \alpha \in V.$$

Obviously, (i) A is symmetric; (ii) an element of A takes a value in $\{0, 1\}$; (iii) a diagonal element of A vanishes. Conversely, for a non-empty set V , a graph structure is uniquely

determined by such a matrix indexed by V . The *degree* or *valency* of a vertex $\alpha \in V$ is defined by

$$\kappa(\alpha) = |\{\beta \in V | \alpha \sim \beta\}|,$$

where $|\cdot|$ denotes the cardinality.

Cayley graphs are defined in terms of a group G and a set R of elements from G , chosen such that the identity element $e \notin R$. Then the Cayley graph of G with respect to R , denoted here by $\Gamma(G, R)$, is defined by [18]:

1. Elements of G are vertices of $\Gamma(G, R)$,
2. For $g, h \in G$, a directed edge from g to h exists if and only if $hg^{-1} \in R$.

Equivalently, this definition is that from any vertex g of a Cayley graph, there are $|R|$ outgoing edges, one to each of the vertices gr , $\forall r \in R$. A Cayley graph will be connected if and only if the set R is a generating set for G , it will be undirected if $r^{-1} \in R, \forall r \in R$. Cayley graphs are always regular, and the degree of a Cayley graph is $|R|$, the cardinality of the generating set.

An automorphism of a graph is a permutation σ on the vertices of Γ so that for every edge $\alpha \sim \beta$ of Γ , $\sigma(\alpha \sim \beta) = \sigma(\alpha) \sim \sigma(\beta)$ is an edge of Γ . Thus, each automorphism of Γ is a one-to-one and onto mapping of the vertices of Γ which preserves adjacency. This implies that an automorphism maps any vertex onto a vertex of the same degree. The set of all such permutations is the automorphism group of the graph which is denoted by $Aut(\Gamma)$. For example, let Γ_1 be the graph in Fig.1, let σ be the permutation (13)(24) and τ be the permutation (123). To see that σ is an automorphism of Γ_1 , notice that the permutation of all the edges are indeed edges of Γ_1 :

$$\sigma(1 \sim 2) = \sigma(1) \sim \sigma(2) = 3 \sim 4$$

$$\sigma(2 \sim 3) = \sigma(2) \sim \sigma(3) = 4 \sim 1$$

$$\sigma(3 \sim 4) = \sigma(3) \sim \sigma(4) = 1 \sim 2$$

$$\sigma(4 \sim 1) = \sigma(4) \sim \sigma(1) = 2 \sim 3$$

$$\sigma(1 \sim 3) = \sigma(1) \sim \sigma(3) = 3 \sim 1.$$

Now to see that τ is not an automorphism of Γ_1 notice that $\tau(1 \sim 4) = \tau(1) \sim \tau(4) = 2 \sim 4 \notin E$.

3 Quotient graph

3.1 Action of an automorphism group and stratification

We can think of letting the automorphism group of a graph act on the set of vertices of the graph. In doing this, we form orbits of vertices. Therefore, now consider a subgroup H of automorphism group $Aut(\Gamma)$. We would like to know what kind of action this subgroup has on the graph and hence on the Hilbert space. First, we define what is meant by the term action [12, 22].

Definition: If X is a set and G is a group, then X is a G set if there is a function $f : G \times X \longrightarrow X$, denoted by $f : (g, x) \longrightarrow gx$, such that:

$$1) ex = x, \forall x \in X,$$

$$2) g(hx) = (gh)x, \forall g, h \in G \text{ and } x \in X.$$

Definition: If X is a G -set and $x \in X$, then the G -orbit (or just orbit) of x is

$$\mathcal{O}(x) = \{gx : g \in G\} \subset X \tag{3-1}$$

The set of orbits of a G -set X form a partition and the orbits correspond to the equivalence classes under the equivalence relation $x \equiv y$ defined by $y = gx$ for some $g \in G$. We can define the action of the subgroup H of the permutation group on the set of basis elements X of the Hilbert space \mathcal{H} as the multiplication of its matrix representation $\sigma(H)$ (in the basis given by the vectors X) with a basis vector. This is a well-defined action since $\sigma(e)|x\rangle = |x\rangle$ and $\sigma(g)(\sigma(h)|x\rangle) = (\sigma(g)\sigma(h))|x\rangle = \sigma(gh)|x\rangle$. Therefore, the set X is partitioned into orbits under the action of H . Since H is a subgroup of the automorphism group, these orbits can be

related to the graph Γ .

Now consider a basis vector $|\alpha\rangle$ and its H -orbit $\mathcal{O}_\alpha = \{\sigma(h)|\alpha\rangle; h \in H\}$. Therefore, due to definition of this orbits, the graph is stratified into a disjoint union of orbits:

$$V = \bigcup_{\alpha \in \Gamma} \mathcal{O}_\alpha. \quad (3-2)$$

With each orbit \mathcal{O}_α we associate a unit vector in $l^2(V)$ defined by

$$|\phi_\alpha\rangle = \frac{1}{\sqrt{|\mathcal{O}_\alpha|}} \sum_{h \in H} \sigma(h)|\alpha\rangle, \quad (3-3)$$

Each of these vectors $|\phi_\alpha\rangle$ are orthonormal, since they are formed from orbits and distinct orbits do not intersect and they span the simultaneous eigenspace of eigenvalue 1 of the matrices $\sigma(H)$ where we denote by $\Lambda(\Gamma)$. Since $\{|\phi_\alpha\rangle\}$ becomes a complete orthonormal basis of $\Lambda(\Gamma)$, we often write

$$\Lambda(\Gamma) = \sum_{\alpha} \oplus \mathbf{C}|\phi_\alpha\rangle. \quad (3-4)$$

Hereafter, we replace the indices of the orbits with integers, i.e., we indicate \mathcal{O}_α and $|\phi_\alpha\rangle$ with \mathcal{O}_i and $|\phi_i\rangle$, respectively ($0 \leq i \leq d$, where d is the numbers of orbits).

Let A be the adjacency matrix of a graph Γ . According to the stratification (3-2), one can obtain a quantum decomposition for the adjacency matrices of this type of graphs as

$$A = A^+ + A^- + A^0. \quad (3-5)$$

where three matrices A^+ , A^- and A^0 are defined as follows: for $\alpha \in \mathcal{O}_i$

$$(A^+)_{\beta\alpha} = \begin{cases} A_{\beta\alpha} & \text{if } \beta \in \mathcal{O}_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

$$(A^-)_{\beta\alpha} = \begin{cases} A_{\beta\alpha} & \text{if } \beta \in \mathcal{O}_{i-1} \\ 0 & \text{otherwise.} \end{cases}$$

$$(A^0)_{\beta\alpha} = \begin{cases} A_{\beta\alpha} & \text{if } \beta \in \mathcal{O}_i \\ 0 & \text{otherwise.} \end{cases}$$

Or, equivalently, for $|\alpha\rangle \in \mathcal{O}_i$,

$$A^+|\alpha\rangle = \sum_{\beta \in \mathcal{O}_{i+1}} |\beta\rangle, \quad A^-|\alpha\rangle = \sum_{\beta \in \mathcal{O}_{i-1}} |\beta\rangle, \quad A^0|\alpha\rangle = \sum_{\beta \in \mathcal{O}_i} |\beta\rangle, \quad (3-6)$$

for $\alpha \sim \beta$. By lemma 2.2, [19] if G is invariant under the quantum components A^ε , $\varepsilon \in \{+, -, 0\}$, then there exist two Szegő- Jacobi sequences $\{\omega_i\}_{i=1}^\infty$ and $\{\alpha_i\}_{i=1}^\infty$ derived from A , such that

$$A^+|\phi_i\rangle = \sqrt{\omega_{i+1}}|\phi_{i+1}\rangle, \quad i \geq 0 \quad (3-7)$$

$$A^-|\phi_0\rangle = 0, \quad A^-|\phi_i\rangle = \sqrt{\omega_i}|\phi_{i-1}\rangle, \quad i \geq 1 \quad (3-8)$$

$$A^0|\phi_i\rangle = \alpha_{i+1}|\phi_i\rangle, \quad i \geq 0, \quad (3-9)$$

where $\sqrt{\omega_i} = \frac{|\mathcal{O}_{i+1}|^{1/2}}{|\mathcal{O}_i|^{1/2}}\kappa_{-(\beta)}$, $\kappa_{-(\beta)} = |\{\alpha \in \mathcal{O}_i | \alpha \sim \beta\}|$ for $\beta \in \mathcal{O}_{i+1}$ and $\alpha_{i+1} = \kappa_{0(\beta)}$, such that $\kappa_{0(\beta)} = |\{\alpha \in \mathcal{O}_i | \alpha \sim \beta\}|$ for $\beta \in \mathcal{O}_i$. In particular $(\Lambda(\Gamma), A^+, A^-)$ is an interacting Fock space associated with a Szegő-Jacobi sequence $\{\omega_i\}$.

3.2 Quotient graphs, spectral distribution of the adjacency matrix graph and CTQW

Based on the action on a graph Γ of the subgroup H of its automorphism group, the quotient graph Γ_H (Γ/H) is that graph whose vertices are the H -orbits, and two such vertices \mathcal{O}_i and \mathcal{O}_j are adjacent in Γ_H if and only if there is an edge in Γ joining a vertex of \mathcal{O}_i to a vertex of \mathcal{O}_j .

With due attention to the idea of calculation of the probability amplitudes for continuous-time quantum walk, in terms of the spectral distribution associated with the adjacency matrix of graphs [13], CTQW on graph Γ induces a CTQW on quotient graph Γ_H as long as adjacency

matrix is as

$$A_{\Gamma_H} = \begin{pmatrix} \alpha_1 & \sqrt{\omega_1} & 0 & \dots & \dots \\ \sqrt{\omega_1} & \alpha_2 & \sqrt{\omega_2} & 0 & \dots \\ 0 & \sqrt{\omega_2} & \alpha_3 & \sqrt{\omega_3} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & \sqrt{\omega_d} & \alpha_d \end{pmatrix}, \quad (3-10)$$

where $\sqrt{\omega_i}$ and α_i are the jumping rate from a vertex to its neighbor and self loops, respectively.

The spectral properties of the adjacency matrix of a graph play an important role in many branches of mathematics and physics . The spectral distribution can be generalized in various ways. In this work, following Refs.[13, 19], we consider the spectral distribution μ of the adjacency matrix A :

$$\langle A^m \rangle = \int_R x^m \mu(dx), \quad m = 0, 1, 2, \dots \quad (3-11)$$

where \langle, \rangle is the mean value with respect to the state $|\phi_0\rangle$. By condition of QD graphs the moment sequence $\{\langle A^m \rangle\}_{m=0}^\infty$ is well-defined[13, 19]. Then the existence of a spectral distribution satisfying (3-11) is a consequence of Hamburgers theorem, see e.g., Shohat and Tamarkin [[21], Theorem 1.2].

We may apply the canonical isomorphism from the interacting Fock space onto the closed linear span of the orthogonal polynomials determined by the Szegő-Jacobi sequences $(\{\omega_i\}, \{\alpha_i\})$. More precisely, the spectral distribution μ under question is characterized by the property of orthogonalizing the polynomials $\{P_n\}$ defined recurrently by

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x - \alpha_1, \\ xP_n(x) &= P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \end{aligned} \quad (3-12)$$

for $n \geq 1$.

As it is shown in [20], the spectral distribution can be determined by the following identity:

$$G_\mu(z) = \int_R \frac{\mu(dx)}{z - x} = \frac{1}{z - \alpha_1 - \frac{\omega_1}{z - \alpha_2 - \frac{\omega_2}{z - \alpha_3 - \frac{\omega_3}{z - \alpha_4 - \dots}}}} = \frac{Q_{n-1}^{(1)}(z)}{P_n(z)} = \sum_{l=1}^n \frac{A_l}{z - x_l}, \quad (3-13)$$

where $G_\mu(x)$ is called the Stieltjes transform and A_l is the coefficient in the Gauss quadrature formula corresponding to the roots x_l of polynomial $P_n(x)$ and where the polynomials $\{Q_n^{(1)}\}$ are defined recurrently as

$$Q_0^{(1)}(x) = 1,$$

$$Q_1^{(1)}(x) = x - \alpha_2,$$

$$xQ_n^{(1)}(x) = Q_{n+1}^{(1)}(x) + \alpha_{n+2}Q_n^{(1)}(x) + \omega_{n+1}Q_{n-1}^{(1)}(x),$$

for $n \geq 1$.

Now if $G_\mu(z)$ is known, then the spectral distribution μ can be recovered from $G_\mu(z)$ by means of the Stieltjes inversion formula:

$$\mu(y) - \mu(x) = -\frac{1}{\pi} \lim_{v \rightarrow 0^+} \int_x^y \text{Im}\{G_\mu(u + iv)\} du. \quad (3-14)$$

Substituting the right hand side of (3-13) in (3-14), the spectral distribution can be determined in terms of $x_l, l = 1, 2, \dots$, the roots of the polynomial $P_n(x)$, and Gauss quadrature constant $A_l, l = 1, 2, \dots$ as

$$\mu = \sum_l A_l \delta(x - x_l) \quad (3-15)$$

(for more details see Refs. [13, 14, 20, 21].)

Due to using the quantum decomposition relations (3-7, 3-8,3-9) and the recursion relation (3-12) of polynomial $P_n(x)$, the other matrix elements $\langle \phi_k | A^m | \phi_0 \rangle$ can be written as

$$\langle \phi_m | A^l | \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2 \cdots \omega_m}} \int_R x^l P_m(x) \mu(dx), \quad l = 0, 1, 2, \dots \quad (3-16)$$

where for obtaining of amplitudes of CTQW in terms of spectral distribution associated with the adjacency matrix of graphs is useful [13].

CTQW on graph were introduced as the quantum mechanical analogue of classical its, which are defined by replacing Kolmogorov's equation (master equation) of continuous-time classical random walk on a graph with Schrödinger's equation. A state $|\phi_0\rangle$ evolves in time as $|\phi(t)\rangle = U(t)|\phi_0\rangle$, where $U(t) = e^{-iHt}$ is the quantum mechanical time evolution operator

(we have set $m = 1$ and $\hbar = 1$). It is natural to choose the Laplacian of the graph, defined as $L = A - D$ as Hamiltonian of walk, where D is a diagonal matrix with entries $D_{jj} = \deg(\alpha_j)$. On d -regular graphs, $D = \frac{1}{d}I$ and since A and D commute, we get

$$e^{-itH} = e^{-it(A - \frac{1}{d}I)} = e^{-it/d} e^{-itA}. \quad (3-17)$$

Hence we can consider $H = A$. Therefore for obtaining the probability amplitude of CTQW at orbit m at time t can be replaced time evolution operator with operator A^t in the equation (3-16) as

$$q_m(t) = \langle \phi_m | e^{-iAt} | \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2 \cdots \omega_m}} \int_R e^{-ixt} P_m(x) \mu(dx). \quad (3-18)$$

The conservation of probability $\sum_{m=0} |q_m(t)|^2 = 1$ follows immediately from Eq.(3-18) by using the completeness relation of orthogonal polynomials $P_n(x)$. Obviously evaluation of $q_m(t)$ leads to the determination of the amplitudes at sites belonging to the stratum V_m , as it is proved in the appendix A [13], the walker has the same amplitude at the vertex belonging to the same stratum, i.e., we have $q_{im}(t) = \frac{q_m(t)}{|V_m|}$, $i = 0, 1, \dots, |V_m|$, where $q_{im}(t)$ denotes the amplitude of the walker at i th vertex of m th stratum.

Formula (3-18) indicates a canonical isomorphism between the interacting Fock space CTQW on QD graphs and the closed linear span of the orthogonal polynomials generated by recursion relations (3-12). This isomorphism was meant to be, a reformulation of CTQW (on QD graphs), which describes quantum states by polynomials (describing quantum state $|\phi_m\rangle$ by $P_m(x)$), and make a correspondence between functions of operators (q -numbers) and functions of classical quantity (c -numbers), such as the correspondence between e^{-iAt} and e^{-ixt} . This isomorphism is similar to the isomorphism between Fock space of annihilation and creation operators a, a^\dagger with space of functions of coherent states parameters in quantum optics, or the isomorphism between Hilbert space of momentum and position operators, and spaces of function defined on phase space in Wigner function formalism.

At the end, by using relation of spectral distribution (3-15) for finite graphs, amplitude of

probability (3-18) is agreeable with

$$q_m(t) = \frac{1}{\sqrt{\omega_1 \omega_2 \cdots \omega_m}} \sum_l A_l e^{-ix_l t} P_m(x_l), \quad (3-19)$$

where by straightforward calculation one can evaluate the average probability for the finite graphs as

$$\bar{P}(m) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |q_m(t)|^2 dt = \frac{1}{\omega_1 \omega_2 \cdots \omega_m} \sum_l A_l^2 P_m^2(x_l). \quad (3-20)$$

4 Examples of quotient graphs

In this section we provide some examples of finite quotient graphs and use the spectral distribution to calculate the relevant amplitudes of continuous-time quantum walks on these graphs.

Example 1. As the first example, consider the Cayley graph $\Gamma(S_3, R)$ where $R = \{r_1, r_2\} = \{(1, 2), (2, 3)\}$. The basis vectors of Hilbert space of walk are $\{|e\rangle, |r_1\rangle, |r_2\rangle, |r_1 r_2\rangle, |r_2 r_1\rangle, |r_2 r_1 r_2\rangle\}$, where the original and the quotient graphs are shown in Fig.2. The automorphism group of this graph is $\text{Aut}(\Gamma(S_3, R)) \simeq R(S_3)$. Consider the subgroup H which corresponds to interchanging the vertices $r_1 \longleftrightarrow r_2$ and $r_1 r_2 \longleftrightarrow r_2 r_1$. Therefore the orbits and unit vectors under the action of this subgroup are,

$$\begin{aligned} \mathcal{O}_0 &= \{|e\rangle\}, & |\phi_0\rangle &= |e\rangle \\ \mathcal{O}_1 &= \{|r_1\rangle, |r_2\rangle\}, & |\phi_1\rangle &= \frac{1}{\sqrt{2}}(|r_1\rangle + |r_2\rangle) \\ \mathcal{O}_2 &= \{|r_1 r_2\rangle, |r_2 r_1\rangle\}, & |\phi_2\rangle &= \frac{1}{\sqrt{2}}(|r_1 r_2\rangle + |r_2 r_1\rangle) \\ \mathcal{O}_3 &= \{|r_2 r_1 r_2\rangle\}, & |\phi_3\rangle &= |r_2 r_1 r_2\rangle. \end{aligned} \quad (4-21)$$

In this case the two Szegő- Jacobi sequences $\{\omega_i\}$ and $\{\alpha_i\}$ are given by

$$\omega_1 = 2, \quad \omega_2 = 1, \quad \omega_3 = 2, \quad \alpha_1 = \alpha_2 = \cdots = 0. \quad (4-22)$$

Hence, The Stieltjes transform and spectral distribution are obtained as

$$G_\mu(z) = \frac{z^3 - 3z}{z^4 - 5z^2 + 4}, \quad \mu(x) = \frac{1}{3}(\delta(x-1) + \delta(x+1)) + \frac{1}{6}(\delta(x-2) + \delta(x+2)). \quad (4-23)$$

Using equations(3-18) or (3-19), we can calculate the probability amplitude of orbits as

$$\begin{aligned} q_0(t) &= \frac{1}{3}(\cos(2t) + 2\cos(t)), \\ q_1(t) &= \frac{-2i}{3\sqrt{2}}(\sin(2t) + \sin(t)), \\ q_2(t) &= \frac{2}{3\sqrt{2}}(\cos(2t) - \cos(t)), \\ q_3(t) &= \frac{i}{3}(-2\sin(2t) + \sin(t)). \end{aligned} \quad (4-24)$$

Example 2. In the second example we consider the Cayley graph $\Gamma(S_3, R)$ where $R = \{r_1, r_2, r_3\} = \{(1, 2), (2, 3), (1, 3)\}$. The original and the quotient graphs are shown in Fig.3. In this case we consider the subgroup H of its automorphism group which corresponds to interchanging the vertices $r_1 \longleftrightarrow r_2 \longleftrightarrow r_3$ and $r_1 r_2 \longleftrightarrow r_1 r_3$. The orbits and unit vectors under the action of this subgroup are,

$$\begin{aligned} \mathcal{O}_0 &= \{|e\rangle\}, \quad |\phi_0\rangle = |e\rangle \\ \mathcal{O}_1 &= \{|r_1\rangle, |r_2\rangle, |r_3\rangle\}, \quad |\phi_1\rangle = \frac{1}{\sqrt{3}}(|r_1\rangle + |r_2\rangle + |r_3\rangle), \\ \mathcal{O}_2 &= \{|r_1 r_2\rangle, |r_2 r_1\rangle\}, \quad |\phi_2\rangle = \frac{1}{\sqrt{2}}(|r_1 r_2\rangle + |r_2 r_1\rangle). \end{aligned} \quad (4-25)$$

Therefore the two Szegő- Jacobi sequences $\{\omega_i\}$, $\{\alpha_i\}$, the Stieltjes transform and spectral distribution are given by

$$\begin{aligned} \omega_1 &= 3, \quad \omega_2 = 6, \quad \alpha_1 = \alpha_2 = \dots = 0. \\ G_\mu(z) &= \frac{z^2 - 6}{z^3 - 9z}, \quad \mu(x) = \frac{2}{3}\delta(x) + \frac{1}{6}(\delta(x-3) + \delta(x+3)). \end{aligned} \quad (4-26)$$

Hence by using equations(3-18) or (3-19), one can calculate the probability amplitude of orbits as

$$q_0(t) = \frac{1}{3}(\cos(3t) + 2),$$

$$\begin{aligned}
q_1(t) &= \frac{-i}{\sqrt{3}} \sin(3t), \\
q_2(t) &= \frac{2}{3\sqrt{2}} (\cos(3t) - 1).
\end{aligned} \tag{4-27}$$

Example 3. In this example we consider the Cayley graph $\Gamma(\mathcal{Z}_n, R)$ with $R = \{1, n-1\}$ which is well known as the cycle graph C_n . The original and quotient graphs are shown in Fig.4. Consider the subgroup H of its automorphism correspond to interchanging the vertices $1 \longleftrightarrow n-1, 2 \longleftrightarrow n-2, \dots k \longleftrightarrow n-k, \dots$. The orbits and unit vectors under the action of this subgroup are

$$\begin{aligned}
\mathcal{O}_0 &= \{|e\rangle\}, \quad |\phi_0\rangle = |e\rangle \\
\mathcal{O}_1 &= \{|1\rangle, |n-1\rangle\}, \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |n-1\rangle), \\
&\vdots \\
\mathcal{O}_{\frac{n-1}{2}} &= \{|\frac{n-1}{2}\rangle, |\frac{n+1}{2}\rangle\}, \quad |\phi_{\frac{n-1}{2}}\rangle = \frac{1}{\sqrt{2}}(|\frac{n-1}{2}\rangle + |\frac{n+1}{2}\rangle),
\end{aligned} \tag{4-28}$$

for the n odd, but if the n is even therefor the last orbit and unit vector is given by

$$\mathcal{O}_{\frac{n}{2}} = \{|\frac{n}{2}\rangle\}, \quad |\phi_{\frac{n}{2}}\rangle = |\frac{n}{2}\rangle. \tag{4-29}$$

Therefore the two Szegő- Jacobi sequences $\{\omega_i\}$, $\{\alpha_i\}$ and spectral distribution are given by:
if n is odd,

$$\omega_1 = 2, \quad \omega_2 = \omega_3 = \dots = \omega_{\frac{n-1}{2}} = 1, \quad \alpha_1 = \alpha_2 = \dots = 0, \quad \alpha_{\frac{n+1}{2}} = 1.$$

$$\mu = \frac{1}{n} (\delta(x-2) + 2 \sum_{l=1}^{\frac{n-1}{2}} \delta(x - 2 \cos(\frac{2l\pi}{n}))), \tag{4-30}$$

if n is even,

$$\omega_1 = 2, \quad \omega_2 = \omega_3 = \dots = \omega_{\frac{n}{2}-1} = 1; \quad \omega_{\frac{n}{2}} = 2; \quad \alpha_1 = \alpha_2 = \dots = 0.$$

$$\mu = \frac{1}{n} (\delta(x-2) + \delta(x+2)) + \frac{2}{n} \sum_{l=1}^{n/2-1} \delta(x - 2 \cos(\frac{2l\pi}{n})). \tag{4-31}$$

Then by using equations(3-18) or (3-19), one can calculate the probability amplitude of orbits which as example we obtain for 0-th orbit as:

if n is odd,

$$q_0(t) = \frac{1}{n}(e^{-it} + 2 \sum_{l=1}^{\frac{n-1}{2}} e^{-it \cos 2l\pi/n}). \quad (4-32)$$

if n is even,

$$q_0(t) = \frac{2}{n}(\cos t + \sum_{l=1}^{n/2-1} e^{-it \cos 2l\pi/n}), \quad (4-33)$$

where the results are in agreement with those of Ref. [13]. In the limit of the large n , the quotient graph is infinite line graph \mathcal{Z} .

5 Conclusion

By using the method of calculation of the probability amplitude for CTQW on graph[13], we have shown CTQW on graph Γ induces a CTQW on quotient graph Γ_H . Then we obtained the probability amplitude of CTQW on some quotient Cayley graphs. In view of the fact that the quotient graphs are important to generate Crystallographic nets, it is possible to generalize this method for investigating CTQW on Crystallographic nets, which is under investigation.

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Figure Captions

Figure-1: The graph Γ_1 .

Figure-2: The graph $\Gamma(S_3, \{(1, 2), (2, 3)\})$ and its quotient graph.

Figure-3: The graph $\Gamma(S_3, \{(1, 2), (2, 3), (1, 3)\})$ and its quotient graph.

Figure-4: The graph $\Gamma(\mathbb{Z}_5, \{1, 4\})$ and its quotient graph